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$210 = 14 \times 15 = 5 \times 6 \times 7 = \binom{21}{2} = \binom{10}{4}$

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Abstract. It is given all the solutions of the diophantine equations

$$(y-1)y(y+1) = \binom{n}{4} \quad \text{and} \quad x(x+1) = \binom{n}{4}.$$

1. Introduction

The title of this paper illustrates the remarkable fact that the number 210 can be represented simultaneously as a product of two consecutive integers, a product of three consecutive integers, a triangular number, and as a binomial coefficient $\binom{n}{4}$ in a nontrivial way[1]. In other words, 210 is a common solution to the system of diophantine equations

$$x(x+1) = (y-1)y(y+1) = \binom{m}{2} = \binom{n}{4}, \tag{1}$$

where we take $x, y, m, n \in \mathbb{Z}$ without further restrictions, i.e. $\binom{m}{2} = \frac{1}{2}m(m-1)$ and $\binom{n}{4} = \frac{1}{24}n(n-1)(n-2)(n-3)$ are defined for all $m, n \in \mathbb{Z}$.

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[1] We prefer not to notice that 210 also is the product of the four smallest prime numbers.

The solution 210 occurs for $x = -15, 14$, $y = 6$, $m = -20, 21$, $n = -7, 10$. There is one other integer that can be represented in the above mentioned four ways: the number 0 occurs for $x = -1, 0$, $y = -1, 0, 1$, $m = 0, 1$, $n = 0, 1, 2, 3$.

In fact, the system (1) consists of six different diophantine equations. We will consider these equations in this paper.

The equation

$$x(x+1) = (y-1)y(y+1)$$

has been solved for the first time in 1963 by MORDELL [M]. It has only the solutions $(x, y) = (-15, 6), (-3, 2), (-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (2, 2), (14, 6)$.

The equation

$$x(x+1) = \binom{m}{2}$$

is essentially a Pell equation, and hence trivial. Its solutions are given by $(x, m) = (x_i, m_i)$ for $i = 0, 1, 2, \ldots$, where $x_{i+1} = 6x_i - x_{i-1} + 2$ and $m_{i+1} = 6m_i - m_{i-1} - 2$, with four different sets of initial values: $(x_0, m_0, x_1, m_1) = (0, 1, 2, 4), (0, 0, 2, -3), (-1, 1, -3, 4), (-1, 0, -3, -3)$.

The equation

$$(y-1)y(y+1) = \binom{m}{2}$$

has been solved for the first time in 1989 by Tzanakis and de WEGER [TW]. It has only the solutions $(y, m) = (-1, 0)$, $(-1, 1)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, -3)$, $(2, 4)$, $(5, -15)$, $(5, 16)$, $(6, -20)$, $(6, 21)$, $(10, -44)$, $(10, 45)$, $(57, -608)$, $(57, 609)$, $(637, -22736)$, $(637, 22737)$.

The equation

$$\binom{m}{2} = \binom{n}{4}$$

has been solved independently by the present two authors, [P] and [dW]. The only solutions are $(m, n) = (-20, -7)$, $(-20, 10)$, $(-5, -3)$, $(-5, 6)$, $(-1, -1)$, $(-1, 4)$, $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, -1)$, $(2, 4)$, $(6, -3)$, $(6, 6)$, $(21, -7)$, $(21, 10)$.

It is the purpose of this note to solve the remaining two equations. We will prove the following two theorems.

Theorem 1. *The equation*

$$(y-1)y(y+1) = \binom{n}{4} \tag{2}$$

has only the solutions $(y,n) = (-1,0), (-1,1), (-1,2), (-1,3), (0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (6,-7), (6,10), (22,-21), (22,24), (26,-24), (26,27)$.

Theorem 2. *The equation*

$$x(x+1) = \binom{n}{4} \tag{3}$$

has only the solutions $(x,n) = (-15,-7), (-15,10), (-1,0), (-1,1), (-1,2), (-1,3), (0,0), (0,1), (0,2), (0,3), (14,-7), (14,10)$.

2. Thue equations for Theorem 1

In equation (2) we put $X = 6y$ and $Y = \frac{3}{4}\left((2n-3)^2 - 5\right)$ (notice that $X, Y \in \mathbb{Z}$). Then equation (2) is seen to be equivalent to

$$Y^2 = X^3 - 36X + 9. \tag{4}$$

This equation defines an elliptic curve, that is of rank 2. We are interested in its integral points, but only in those with $6 \mid X$.

Let $\mathbb{K} = \mathbb{Q}(\theta)$, where θ is a root of $X^3 - 36X + 9$. Then an integral basis of $\mathbb{K}$ is $\{1, \theta, \frac{1}{3}\theta^2\}$, the class group is C_3, a system of fundamental units is

$$\epsilon = 1 - 4\theta - 2\frac{1}{3}\theta^2, \quad \eta = 1 - 4\theta + 2\frac{1}{3}\theta^2.$$

The ramifying primes are $3, 11$ and 23, and they ramify as follows:

$$\langle 3 \rangle = \mathfrak{p}_3^3, \quad \mathfrak{p}_3 = \left\langle -12 + \frac{1}{3}\theta^2 \right\rangle, \quad \langle 11 \rangle = \mathfrak{p}_{11}^2 \mathfrak{q}_{11}, \quad \langle 23 \rangle = \mathfrak{p}_{23}^2 \mathfrak{q}_{23},$$

where $\mathfrak{p}_{11}, \mathfrak{q}_{11}, \mathfrak{p}_{23}, \mathfrak{q}_{23}$ are non-principal prime ideals. Note that

$$X^3 - 36X + 9 = (X - \theta)\left(X^2 + \theta X + (\theta^2 - 36)\right),$$

and if a prime ideal $\mathfrak{p}$ divides both $\langle X - \theta \rangle$ and $\left\langle X^2 + \theta X + (\theta^2 - 36) \right\rangle$, then it divides $\left\langle (X + 2\theta)(X - \theta) - \left(X^2 + \theta X + (\theta^2 - 36)\right) \right\rangle = \left\langle 3^2 \left(-4+\right.\right.$

$\frac{1}{3}\theta^2)\rangle = \mathfrak{p}_3^6\mathfrak{p}_{11}^2\mathfrak{p}_{23}^2$. Since $3 \mid X$ and $\operatorname{ord}_{\mathfrak{p}_3}(\theta) = 2$, we have $\operatorname{ord}_{\mathfrak{p}_3}(X-\theta) = 2$, and $\operatorname{ord}_{\mathfrak{p}_3}\left(X^2 + \theta X + (\theta^2 - 36)\right) = 4$. Thus from equation (4) we see that there are $a, b \in \{0,1\}$ and an integral ideal $\mathfrak{a}$ such that

$$\langle X - \theta\rangle = \mathfrak{p}_3^2\mathfrak{p}_{11}^a\mathfrak{p}_{23}^b\mathfrak{a}^2.$$

On taking norms we find $Y^2 = 3^2 11^a 23^b (N\mathfrak{a})^2$, so that $a = b = 0$. Further it follows that $\mathfrak{a}^2$ is principal, hence so is $\mathfrak{a}$. There exist $m, n \in \{0,1\}$ such that

$$X - \theta = \pm\epsilon^m\eta^n\left(-12 + \tfrac{1}{3}\theta^2\right)^2\alpha^2,$$

where α is a generator of $\mathfrak{a}$.

Now we look at embeddings of $\mathbb{K}$ into $\mathbb{R}$. We write $\theta_1 = -6.12\ldots$, $\theta_2 = 0.25\ldots$, $\theta_3 = 5.87\ldots$, and then find that ϵ_2 and ϵ_3 are negative, whereas ϵ_1 and all conjugates of η are positive. Comparing norms, using that $N(X - \theta) = Y^2 > 0$ and $N\epsilon = N\eta = 1$, we see that the $\pm$-sign in (5) is $+$. Further, if $X \geq 6$ then $X - \theta_i > 0$ for $i = 1,2,3$, and it follows by studying the signs that $m = 0$. Notice that the solutions of (4) with $X < 6$ (and $6 \mid X$) are trivially found to be only $X = -6, 0$, leading to $Y = \pm 3$ in both cases, and further to $(y,n) = (-1,0), (-1,1), (-1,2), (-1,3), (0,0), (0,1), (0,2), (0,3)$.

2.1. The case $n = 0$

In (5) we now may put $\alpha = A + B\theta + C\frac{1}{3}\theta^2$, and if $n = 0$ we then find

$$X - \theta = \left(-12 + \frac{1}{3}\theta^2\right)^2\left(A + B\theta + C\frac{1}{3}\theta^2\right)^2.$$

Expanding out and comparing coefficients, we obtain

$$X = 144A^2 + 72AB + 6AC + 9B^2, \tag{6}$$

$$1 = A^2 - 6BC, \tag{7}$$

$$0 = 4A^2 + 2AB - C^2. \tag{8}$$

Equation (7) implies that A is odd, and that A and B are coprime. Thus A and $2A+B$ are coprime, and equation (8), written as $C^2 = 2A(2A+B)$, is seen to imply the existence of $E, F \in \mathbb{Z}$ with

$$A = E^2, \quad B = 2F^2 - 2E^2, \quad C = 2EF.$$

Substituting these expressions into (7) we have

$$E^4 + 24E^3F - 24EF^3 = E(E^3 + 24E^2F - 24F^3) = 1.$$

Clearly $E = E^3 + 24E^2F - 24F^3 = \pm 1$, hence this is trivial: the only solutions are given by $(E, F) = \pm(1,-1), \pm(1,0), \pm(1,1)$, leading respectively to $(A, B, C) = (1,0,-2), (1,0,2), (1,-2,0)$, and further to $(X, Y) = (132, \pm 1515)$, $(36, \pm 213)$, $(156, \pm 1947)$, and finally to $(y, n) = (22, -21)$, $(22, 24)$, $(6, -7)$, $(6, 10)$, $(26, -24)$, $(26, 27)$.

2.2. The case $n = 1$

In (5) we again put $\alpha = A + B\theta + C\frac{1}{3}\theta^2$, and if $n = 1$ we then find by $1/\eta = 25 - 2\frac{1}{3}\theta^2$ that

$$\left(25 - 2\frac{1}{3}\theta^2\right)(X - \theta) = \left(-12 + \frac{1}{3}\theta^2\right)^2 \left(A + B\theta + C\frac{1}{3}\theta^2\right)^2.$$

Expanding out and comparing coefficients, we obtain

$$25X - 6 = 144A^2 + 72AB + 6AC + 9B^2, \tag{9}$$

$$1 = A^2 - 6BC, \tag{10}$$

$$\frac{2}{3}X = 4A^2 + 2AB - C^2. \tag{11}$$

Now $2 \times (9) + 12 \times (10) - 75 \times (11)$ gives

$$25C^2 + (4A - 24B)C + (-2AB + 6B^2) = 0.$$

We view this equation as a quadratic equation in C. If it is to have rational solutions, the discriminant must be a square, D^2 say. Hence

$$D^2 = (4A - 24B)^2 - 100(-2AB + 6B^2) = 8(A - B)(2A + 3B).$$

If p is a prime dividing both $A - B$ and $2A + 3B$, then it divides $5A$ and $5B$, and since A and B are coprime, it must be 5. It follows that we can write

$$A - B = eE^2, \quad 2A + 3B = fF^2$$

for unknown integers E, F, where for (e, f) we have four cases:

$$(e,f) = (1,2), (2,1), (5,10), (10,5).$$

So we get

$$A = \frac{3}{5}eE^2 + \frac{1}{5}fF^2, \quad B = -\frac{2}{5}eE^2 + \frac{1}{5}fF^2,$$
$$C = -\frac{6}{25}eE^2 \pm \frac{1}{25}\sqrt{2ef}EF + \frac{2}{25}fF^2, \quad D = 2\sqrt{2ef}EF.$$

Since F is defined up to sign, we can replace the $\pm$ sign by a $+$. Now we substitute the above expressions into equation (10), and find

$$-27e^2E^4 + 12e\sqrt{2ef}E^3F + 90efE^2F^2 - 6f\sqrt{2ef}EF^3 - 7f^2F^4 = 125.$$

On putting $U = 5\sqrt{2e/f}E$, $V = \sqrt{2e/f}E - F$, which are both integers, we get the Thue equation

$$U^4 - 8U^3V - 12U^2V^2 + 136UV^3 - 140V^4 = \frac{2500}{f^2}.$$

Notice that with $f = 1, 2, 5, 10$ we have $\frac{2500}{f^2} = 2500, 625, 100, 25$. The following Theorem treats these Thue equations. Its proof is postponed to a forthcoming section.

Theorem 3. *The Thue equations*

$$f_1(U,V) = U^4 - 8U^3V - 12U^2V^2 + 136UV^3 - 140V^4 = m, \quad m \in \{25, 100, 625, 2500\} \tag{12}$$

have only the solutions $(U,V) = \pm(3,1)$ *at* $m = 25$, *and* $(U,V) = \pm(5,0), \pm(5,2)$ *at* $m = 625$.

The solutions $(U,V) = \pm(3,1)$ lead to $(e,f) = (5,10)$, and to non-integral E, F. The solutions $(U,V) = \pm(5,0)$ lead to $(e,f) = (1,2)$, $(E,F) = \pm(1,1)$, $(A,B,C) = (1,0,0)$, $(X,Y) = (6,\pm 3)$, and finally to $(y,n) = (1,0), (1,1), (1,2), (1,3)$. The solutions $(U,V) = \pm(5,2)$ lead to $(e,f) = (1,2)$, $(E,F) = \pm(1,-1)$, and then to non-integral C.

This completes the proof of Theorem 1.

3. Thue equations for Theorem 2

In equation (3) we put $X = 2n - 3$ and $Y = 8x + 4$. Then equation (3) is seen to be equivalent to

(13) $$6Y^2 = X^4 - 10X^2 + 105.$$

This equation defines an elliptic curve, that is of rank 2. We are interested in its integral points.

The right hand side of (13) can be written as

$$(X^2 - 5)^2 + 80 = \left(X^2 - 5 + 4\sqrt{-5}\right)\left(X^2 - 5 - 4\sqrt{-5}\right).$$

Let $\mathbb{K} = \mathbb{Q}\left(\sqrt{-5}\right)$. The class group is C_2, and we need to know the behaviour of the primes 2, 3 and 5, which is as follows:

$$\langle 2 \rangle = \mathfrak{p}_2^2, \quad \langle 3 \rangle = \mathfrak{p}_3\overline{\mathfrak{p}}_3, \quad \langle 5 \rangle = \mathfrak{p}_5^2, \quad \mathfrak{p}_5 = \left\langle \sqrt{-5} \right\rangle,$$

where $\mathfrak{p}_2, \mathfrak{p}_3$ are non-principal ideals, the bar denotes complex conjugation, and we have the relations

$$\overline{\mathfrak{p}}_2 = \mathfrak{p}_2, \quad \mathfrak{p}_2\mathfrak{p}_3 = \left\langle 1 + \sqrt{-5} \right\rangle, \quad \mathfrak{p}_3^2 = \left\langle 2 - \sqrt{-5} \right\rangle.$$

If $\mathfrak{p}$ is a prime ideal dividing both $\left\langle X^2 - 5 + \sqrt{-5} \right\rangle$ and $\left\langle X^2 - 5 - 4\sqrt{-5} \right\rangle$, then it divides $\left\langle \left(X^2 - 5 + 4\sqrt{-5}\right) - \left(X^2 - 5 - 4\sqrt{-5}\right) \right\rangle = \left\langle 8\sqrt{-5} \right\rangle = \mathfrak{p}_2^6\mathfrak{p}_5$. It follows by (13) that there exist $a, b, c, d \in \{0, 1\}$ and an integral ideal $\mathfrak{a}$ such that

$$\left\langle X^2 - 5 + 4\sqrt{-5} \right\rangle = \mathfrak{p}_2^a \mathfrak{p}_3^b \overline{\mathfrak{p}_3}^c \mathfrak{p}_5^d \mathfrak{a}^2.$$

Taking norms we have $6Y^2 = 2^a 3^{b+c} 5^d (N\mathfrak{a})^2$, hence $a = 1$, $(b, c) = (1, 0)$ or $(0, 1)$, $d = 0$. Notice that $\mathrm{ord}_{\mathfrak{p}_2}(X^2 - 1) \geq 6$, and $\mathrm{ord}_{\mathfrak{p}_2}\left(-4 + 4\sqrt{-5}\right) = 5$, so that we find $\mathrm{ord}_{\mathfrak{p}_2}(\mathfrak{a}) = 2$. Hence if $\mathfrak{a}$ is principal we may write $\mathfrak{a} = \left\langle 2A + 2B\sqrt{-5} \right\rangle$, and if $\mathfrak{a}$ is non-principal, then $\mathfrak{a}/\mathfrak{p}_2$ is principal, and we may write $\mathfrak{a} = \mathfrak{p}_2 \left\langle A + B\sqrt{-5} \right\rangle$, where in both cases $A, B \in \mathbb{Z}$. We define $p = 0$ if $\mathfrak{a}$ is principal, and $p = 1$ if $\mathfrak{a}$ is non-principal. Then $\mathfrak{a}^2 = 2^{2-p} \left\langle A^2 - 5B^2 + 2AB\sqrt{-5} \right\rangle$.

3.1. The case $(b,c)=(1,0)$

In the case $(b,c)=(1,0)$, going from ideals to generators, we thus have

$$\pm 2^p\left(\frac{X^2-5}{4}+\sqrt{-5}\right)=\left(1+\sqrt{-5}\right)\left(A^2-5B^2+2AB\sqrt{-5}\right).$$

Comparing real and imaginary parts we get

$$\pm 2^p\frac{X^2-5}{4}=A^2-10AB-5B^2, \tag{14}$$

$$\pm 2^p=A^2+2AB-5B^2. \tag{15}$$

Then $4\times(14)+5\times(15)$ yields

$$2^pX^2=9A^2-30AB-45B^2=(3A-5B)^2-70B^2.$$

Thus the next field to study is $\mathbb{L}=\mathbb{Q}\left(\sqrt{70}\right)$. Its class group is C_2, a fundamental unit is $251+30\sqrt{70}$, and the primes $2,3,5$ and 7 behave as follows:

$$\langle 2\rangle=\mathfrak{p}_2^2,\quad \langle 3\rangle=\mathfrak{p}_3\mathfrak{q}_3,\quad \langle 5\rangle=\mathfrak{p}_5^2,\quad \mathfrak{p}_5=\left\langle 25+3\sqrt{70}\right\rangle,\quad \langle 7\rangle=\mathfrak{p}_7^2,$$

where $\mathfrak{p}_2,\mathfrak{p}_3,\mathfrak{q}_3,\mathfrak{p}_7$ are non-principal prime ideals. If $\mathfrak{p}$ is a prime ideal dividing both
$\langle 3A-5B+B\sqrt{70}\rangle$ and $\langle 3A-5B-B\sqrt{70}\rangle$, then it divides
$\langle(3A-5B+B\sqrt{70})+(3A-5B-B\sqrt{70})\rangle=\langle 2(3A-5B)\rangle$ and also
$\langle(3A-5B+B\sqrt{70})-(3A-5B-B\sqrt{70})\rangle=\langle 2B\sqrt{70}\rangle$.
Since A and B are relatively prime (by (15)) we find that $\mathfrak{p}$ divides $2,3,5$ or 7. It follows that there exist $a,b,c,d,e\in\{0,1\}$ and an integral ideal $\mathfrak{b}$ such that

$$\left\langle 3A-5B+B\sqrt{70}\right\rangle=\mathfrak{p}_2^a\mathfrak{p}_3^b\mathfrak{q}_3^c\mathfrak{p}_5^d\mathfrak{p}_7^e\mathfrak{b}^2.$$

Taking norms we find that $2^pX^2=2^a3^{b+c}5^d7^e(N\mathfrak{b})^2$, and thus that $a=p=0$ or 1, $b=c=0$ or 1, $d=e=0$. Since $\langle 3A-5B+B\sqrt{70}\rangle$, $\mathfrak{p}_3\mathfrak{q}_3$ and $\mathfrak{b}^2$ are principal ideals, it follows that $a=p=0$. Then it also follows that in (14) and (15) the $\pm$ sign is a $+$, because $A^2+2AB-5B^2=-1$ has no solutions.

If $\mathfrak{b}$ is principal, we may write $\mathfrak{b} = \langle E + F\sqrt{70}\rangle$, and if $\mathfrak{b}$ is non-principal, then $\mathfrak{b}\mathfrak{p}_2$ is principal, and we may write $\mathfrak{b}\mathfrak{p}_2 = \langle E + F\sqrt{70}\rangle$, where in both cases E, F are unknown integers. We let $q = 0$ if $\mathfrak{b}$ is principal, and $q = 1$ if $\mathfrak{b}$ is non-principal. Then, going from ideals to generators, we can write

$$\pm 2^q \left(3A - 5B + B\sqrt{70}\right) = \left(251 + 30\sqrt{70}\right)^n 3^b \left(E^2 + 70F^2 + 2EF\sqrt{70}\right),$$

where also n can be taken to be in $\{0, 1\}$. As A and B are defined up to sign, we may take the $\pm$ sign to be a $+$.

3.1.1. The case $n = 0$

In the case $n = 0$, writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we obtain

$$3A - 5B = e(E^2 + 70F^2),$$
$$B = 2eEF,$$

hence

$$A = \frac{1}{3}e(E^2 + 10EF + 70F^2).$$

We substitute these expressions into (15), and thus get

$$E^4 + 32E^3F + 180E^2F^2 + 2240EF^3$$
$$+4900F^4 = \frac{9}{e^2}.$$

We prefer to substitute $E = U - 2V, F = V$, to get somewhat smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives the Thue equations

$$U^4 + 24U^3V + 12U^2V^2 + 1872UV^3 + 900V^4 = m \tag{16}$$

for $m = \frac{9}{e^2} \in \{1, 4, 9, 36\}$. Below we will treat these Thue equations.

3.1.2. The case $n = 1$

In the case $n = 1$, again writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we find

$$3A - 5B = e(251E^2 + 4200EF + 17570F^2),$$
$$B = e(30E^2 + 502EF + 2100F^2),$$

hence

$$A = \frac{1}{3}e(401E^2 + 6710EF + 28070F^2).$$

We substitute these expressions into (15), and thus get

$$192481E^4 + 6441632E^3F + 80841780E^2F^2 + 450914240EF^3$$
$$+943156900F^4 = \frac{9}{e^2}.$$

We prefer to substitute $E = 3U - 31V, F = -\frac{5}{14}U + \frac{26}{7}V$, to get much smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives in fact the Thue equations (16), but this time with $m = \frac{1764}{e^2} \in \{196, 784, 1764, 7056\}$.

In a forthcoming section we will prove the following result.

Theorem 4. *The Thue equations*

$$f_2(U,V) = U^4 + 24U^3V + 12U^2V^2 + 1872UV^3 + 900V^4 = m, \quad m \in \{1,4,9,36,196,784,1764,7056\} \tag{17}$$

have only the solutions $(U, V) = \pm(1, 0)$ *at* $m = 1$.

The solutions $(U, V) = \pm(1, 0)$ lead to $m = 1$, $n = 0$, $e = 3$, $(E, F) = \pm(1, 0)$, $(A, B) = (1, 0)$, $(X, Y) = (\pm 3, \pm 4)$, and finally to $(x, n) = (-1, 0), (-1, 3), (0, 0), (0, 3)$.

3.2. The case $(b, c) = (0, 1)$

In the case $(b, c) = (0, 1)$, going from ideals to generators, we have

$$\pm 2^p \left(\frac{X^2 - 5}{4} + \sqrt{-5}\right) = \left(1 - \sqrt{-5}\right)\left(A^2 - 5B^2 + 2AB\sqrt{-5}\right).$$

Comparing real and imaginary parts we get

$$\pm 2^p \frac{X^2 - 5}{4} = A^2 + 10AB - 5B^2, \tag{18}$$

$$\mp 2^p = A^2 - 2AB - 5B^2. \tag{19}$$

Then $4 \times (18) - 5 \times (19)$ yields

$$\mp 2^p X^2 = A^2 - 50AB - 5B^2 = (A - 25B)^2 - 630B^2.$$

Again we work in $\mathbb{L} = \mathbb{Q}\left(\sqrt{70}\right)$. If $\mathfrak{p}$ is a prime ideal dividing both $\left\langle A - 25B + 3B\sqrt{70}\right\rangle$ and $\left\langle A - 25B - 3B\sqrt{70}\right\rangle$, then as above we see that $\mathfrak{p}$ divides $2, 3, 5$ or 7. It follows that there exist $a, b, c, d, e \in \{0, 1\}$ and an integral ideal $\mathfrak{b}$ such that

$$\left\langle A - 25B + 3B\sqrt{70}\right\rangle = \mathfrak{p}_2^a \mathfrak{p}_3^b \mathfrak{q}_3^c \mathfrak{p}_5^d \mathfrak{p}_7^e \mathfrak{b}^2.$$

Taking norms we find that $2^p X^2 = 2^a 3^{b+c} 5^d 7^e (N\mathfrak{b})^2$, and thus that $a = p = 0$ or 1, $b = c = 0$ or 1, $d = e = 0$. Since $\left\langle 3A - 5B + B\sqrt{70}\right\rangle$, $\mathfrak{p}_3\mathfrak{q}_3$ and $\mathfrak{b}^2$ are principal ideals, it follows that $a = p = 0$. Then it also follows that in (18) and (19) the $\pm$ and $\mp$ signs respectively are $-$ and $+$, because $A^2 - 2AB - 5B^2 = -1$ has no solutions.

If $\mathfrak{b}$ is principal, we may write $\mathfrak{b} = \left\langle E + F\sqrt{70}\right\rangle$, and if $\mathfrak{b}$ is non-principal, then $\mathfrak{b}\mathfrak{p}_2$ is principal, and we may write $\mathfrak{b}\mathfrak{p}_2 = \left\langle E + F\sqrt{70}\right\rangle$, where in both cases E, F are unknown integers. We let $q = 0$ if $\mathfrak{b}$ is principal, and $q = 1$ if $\mathfrak{b}$ is non-principal. Then, going from ideals to generators, we can write

$$\begin{aligned} &\pm 2^q \left(A - 25B + 3B\sqrt{70}\right) \\ &= \left(251 + 30\sqrt{70}\right)^n 3^b \left(E^2 + 70F^2 + 2EF\sqrt{70}\right), \end{aligned}$$

where also n can be taken to be in $\{0, 1\}$. As A and B are defined up to sign, we may take the $\pm$ sign to be a $+$.

3.2.1. *The case $n = 0$*

In the case $n = 0$, writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we obtain

$$A - 25B = e(E^2 + 70F^2), \quad 3B = 2eEF,$$

hence

$$eA = \frac{1}{3}e(3E^2 + 50EF + 210F^2), \quad B = \frac{2}{3}eEF.$$

We substitute these expressions into (19), and thus get

$$E^4 + 32E^3F + \frac{1180}{3}E^2F^2 + 2240EF^3 + 4900F^4 = \frac{1}{e^2}.$$

We prefer to substitute $E = \frac{1}{3}U - \frac{19}{3}V, F = V$, to get somewhat smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives the Thue equations

$$U^4 + 20U^3V + 234U^2V^2 + 2492UV^3 - 2423V^4 = m \tag{20}$$

for $m = \frac{81}{e^2} \in \{9, 36, 81, 324\}$. Below we will treat these Thue equations.

3.2.2. The case $n = 1$

In the case $n = 1$, again writing $e = 2^{-q}3^b$ (thus $e \in \{1, 3, \frac{1}{2}, \frac{3}{2}\}$), and comparing coefficients, we find

$$\begin{aligned} A - 25B &= e(251E^2 + 4200EF + 17570F^2), \\ 3B &= e(30E^2 + 502EF + 2100F^2), \end{aligned}$$

hence

$$\begin{aligned} A &= \frac{1}{3}e(1503E^2 + 25150EF + 105210F^2), \\ B &= \frac{1}{3}e(30E^2 + 502EF + 2100F^2). \end{aligned}$$

We substitute these expressions into (19), and thus get

$$\begin{aligned} 240481E^4 + 8048032E^3F + \frac{303005980}{3}E^2F^2 \\ +563362240EF^3 + 1178356900F^4 = \frac{1}{e^2}. \end{aligned}$$

We prefer to substitute $E = \frac{5}{3}U - \frac{221}{3}V, F = -\frac{1}{5}U + \frac{44}{5}V$, to get much smaller coefficients. Notice that $U, V \in \mathbb{Z}$. This gives in fact the Thue equations (20), but this time with $m = \frac{2025}{e^2} \in \{225, 900, 2025, 8100\}$.

In a forthcoming section we will prove the following result.

Theorem 5. *The Thue equations*

$$\begin{aligned} f_3(U, V) = U^4 + 20U^3V + 234U^2V^2 + 2492UV^3 - 2423V^4 = m, \\ m \in \{9, 36, 81, 225, 324, 900, 2025, 8100\} \end{aligned} \tag{21}$$

have only the solutions $(U,V) = \pm(3,0)$ *at* $m = 81$, *and* $(U,V) = \pm(1,1)$ *at* $m = 324$, *and* $(U,V) = \pm(17,-1)$ *at* $m = 8100$.

The solutions $(U,V) = \pm(3,0)$ lead to $m = 81$, $e = 1$, $n = 0$, $(E,F) = \pm(1,0)$, $(A,B) = (1,0)$, $(X,Y) = (\pm 1, \pm 4)$, and finally to $(x,n) = (-1,1), (-1,2), (0,1), (0,2)$. The solutions $(U,V) = \pm(1,1)$ lead to $m = 324$, $e = \frac{1}{2}$, $n = 0$, $(E,F) = \pm(-6,1)$, $(A,B) = (3,-2)$, $(X,Y) = (\pm 17, \pm 116)$, and finally to $(x,n) = (-15,-7), (-15,10), (14,-7), (14,10)$. The solutions $(U,V) = \pm(17,-1)$ lead to $m = 8100$, $e = \frac{1}{2}$, $n = 1$, and then to non-integral F. This completes the proof of Theorem 2.

4. Solving the Thue equations

In this section we finally prove Theorems 3, 4 and 5, thus completing also the proofs of Theorems 1 and 2. Using the program package KANT (PC-DOS version) we obtain the following results:

Equation	*Solutions*	*486PC-CPU-time (sec)*
$f_1(x,y)=25$	$(-3,-1),(3,1)$	38
$f_1(x,y)=100$	–	33
$f_1(x,y)=625$	$(-5,-2),(-5,0),(5,0),(5,2)$	71
$f_1(x,y)=2500$	–	110
$f_2(x,y)=1$	$(-1,0),(1,0)$	15
$f_2(x,y)=4$	–	9
$f_2(x,y)=9$	–	9
$f_2(x,y)=36$	–	10
$f_2(x,y)=196$	–	10
$f_2(x,y)=784$	–	18
$f_2(x,y)=1764$	–	28
$f_2(x,y)=7056$	–	23
$f_3(x,y)=9$	–	15
$f_3(x,y)=36$	–	10
$f_3(x,y)=81$	$(-3,0),(3,0)$	23
$f_3(x,y)=225$	–	29
$f_3(x,y)=324$	$(-1,-1),(1,1)$	45
$f_3(x,y)=900$	–	36
$f_3(x,y)=2025$	–	60
$f_3(x,y)=8100$	$(-17,1),(17,-1)$	198

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